On the main quantum number dependence of the pionium production

R. Lednický

Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia
Institute of Physics ASCR, Na Slovance 2, 18221 Prague 8, Czech Republic

We will use here the notation and equations from Ref. [1]. Neglecting the production of the $\pi^+\pi^-$ atoms with the orbital angular momentum $l > 0$ (suppressed by powers of the $\pi^+\pi^-$ Bohr radius $|a| = 387.5$ fm), the probability to produce a pionium as a result of the two-pion final state interaction (FSI) depends on the main quantum number $n$ as (see Eq. (28) in Ref. [1])

$$w_n \propto (1 + \delta_n)|\psi_{n0}^{\text{conf}}(0)|^2 \propto (1 + \delta_n)/n^3,$$

where $\psi_{n0}^{\text{conf}}(0)$ is the pure Coulomb wave function of the $\pi^+\pi^-$ atom at zero separation and the correction factor $(1 + \delta_n)$ takes into account the effects of finite-size of the pion production region and the two-pion strong FSI. It can be shown that for two pions produced at a distance in their center-of-mass system much smaller than the Bohr radius $|a|$, the $n$-dependence of the correction factor is dominated by the renormalization effect of the strong FSI on the two-pion atomic wave function (see Eq. (126) in Ref. [1]):

$$(1 + \delta_n) = \left[1 + \phi(n)\frac{2RA^\alpha}{n|a|}\right](1 + \delta_n').$$

Here $\phi(n) \approx 3$ and $RA^\alpha \approx 0.2$ fm are respectively defined in Eqs. (80), (86) and (115) of Ref. [1]. Thus

$$\phi(n) = 2 + 2n[\ln n - \psi(n)],$$

where the digamma function $\psi(n)$ for the integer argument is given by the recurrence relation:

$$\psi(n + 1) = \psi(n) + 1/n, \quad \psi(1) = -\gamma = -0.5772156649.$$

With the increasing $n$, $\phi(n)$ slowly converges to 3, the first 10 values to 5 digit accuracy being equal to 3.15443, 3.08145, 3.05497, 3.04141, 3.03320, 3.02770, 3.02376, 3.02080, 3.01850, 3.01665. Further,

$$RA^\alpha \approx f_0^{\alpha\alpha} - f_0^{\beta\beta} - f_0^{\alpha\beta} (k_{f_0}^*)^2 \approx f_0^{\alpha\alpha} = f_0,$$

where the amplitudes $f_0^{\alpha\alpha}$ are expressed through the two-pion isoscalar and isotensor s-wave scattering lengths $a_0^0$ and $a_0^2$ as (see Eq. (108) in Ref. [1])

$$f_0^{\alpha\alpha} = \frac{2}{3}a_0^0 + \frac{1}{3}a_0^2, \quad f_0^{\alpha\beta} = f_0^{\beta\alpha} = \frac{-\sqrt{2}}{3}(a_0^0 - a_0^2), \quad f_0^{\beta\beta} = \frac{1}{3}a_0^0 + \frac{2}{3}a_0^2.$$
and $k^2_{\beta} = 35.5$ MeV/c is the $\pi^0$ momentum in the channel $\beta = \{\pi^0\pi^0\}$ at the threshold of the channel $\alpha = \{\pi^+\pi^-\}$. Using the values of the scattering lengths from Ref. [2], one has $\mathcal{R}A^{\alpha\alpha} = 0.18635$ fm.

Taking into account that the factor $(1 + \delta'_n)$ is practically independent of $n$ except for a tiny fraction of the pairs containing a pion from $\eta'$ decay (see the most right panel in Fig. 12 of Ref. [1]), one can write the $n$-dependence of the pionium production probability in a simple analytical form:

$$w_n \propto \left[ 1 + \phi(n) \frac{2\mathcal{R}A^{\alpha\alpha}}{|a|} \right] \frac{1}{n^3} \approx \left( 1 + \frac{0.3\%}{n} \right) \frac{1}{n^3},$$

where the approximate equality neglects a weak $n$-dependence of $\phi(n)$.

In Refs. [3, 4], the effect of the strong interaction on the $n$–dependence of the pionium wave function has been studied numerically, solving the corresponding Schrödinger equations. Thus, in Ref. [3], the ratio $R_n = \psi_{\text{chiral}}^0/\psi_{\text{std}}^0$ and the difference $\Delta R_n = R_1 - R_n$ have been calculated for $n = 1 - 3$ using an exponential form of the short-range potential. According to Eqs. (83), (90) and (92) of Ref. [1], one has, up to corrections $\mathcal{O}(f_0/a)$ and $\mathcal{O}(r^2/a^2)$:

$$R_n \equiv \frac{\psi_{\text{chiral}}^0(r^*)}{\psi_{\text{std}}^0(r^*)} \approx 1 + \frac{f_0}{r^*}, \quad \Delta R_n \equiv R_1 - R_n = \frac{f_0}{|a|} \left\{ \phi(1) - \frac{1}{n} \phi(n) \right\} \left( 1 + \frac{f_0}{r^*} \right).$$

From Fig. 1 of Ref. [3], one can deduce a value of $\sim 0.15$ fm for the scattering length $f_0$ to achieve an agreement with the prediction of Eq. (8) for the ratio $R_n$ at $d < r^* \ll |a|$. The differences $\Delta R_n$, presented in Fig. 1 of Ref. [3] for $n = 2$ and 3, are however by a factor 1.6 higher than the corresponding predictions of Eq. (8). For example, for $10^3 \Delta R_n$ at $r^* = 8$ fm, $n = 2$ and 3, one can read from this figure the values $1.0$ and $1.3$ while, Eq. (8) respectively predicts $0.6$ and $0.8$. This discrepancy may indicate that the calculation error, declared in Ref. [3] to be better than $10^{-4}$, was underestimated by a factor of 5.

In Ref. [4], a more refined numerical study of the $n$–dependence has been done accounting for the second channel ($\pi^0\pi^0$) and extended charges. The hadronic $\pi\pi$ potentials have been chosen to reproduce the phase shifts given by two–loop chiral perturbation theory. The quantity $d_n = n^{3/2} \psi_{\text{chiral}}^0/\psi_{\text{std}}^0 - 1$ has been calculated for $n = 1 - 4$. Similar to Eq. (8), one has for $d < r^* \ll |a|$:

$$d_n \equiv n^{3/2} \frac{\psi_{\text{chiral}}^0(r^*)}{\psi_{\text{std}}^0(r^*)} - 1 = \frac{f_0}{|a|} \left\{ \phi(1) - \frac{1}{n} \phi(n) \right\},$$

up to corrections $\mathcal{O}(f_0 r^*/a^2)$ and $\mathcal{O}(r^2/a^2)$. The results of numerical calculations presented in Fig. 2 of Ref. [4] are in qualitative agreement with Eq. (9), $d_n$ being almost constant (except for the region of very small $r^*$) and showing the right $n$–dependence: $d_n \sim - (1 - 1/n)$. Similar to Ref. [3], the numerical results for $d_n$ are however higher, now by a factor of 2.5, than the predictions of Eq. (9) calculated with $f_0 = 0.2$ fm which should correspond within $\sim 10\%$ to the choice of the potentials in Ref. [4]. Since the presence of the second channel leads to a negligible modification of Eq. (9) ($\mathcal{R}A^{\alpha\alpha} \approx f_0$) and the correction due to the extended charges is also expected to be negligible ($\sim \frac{1}{2f_0^2} r^2/a^2$), the discrepancy in the size of the correction $d_n$ has to be attributed to the insufficient calculation accuracy or, to the incorrect matching of the scattering length.

\(^{1}\)One should correct the figure by interchanging the curves. The author is grateful to O. Voskresenskaya for pointing out this misprint.
References


